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Complementary variational principles for a class of biharmonic problems

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SUMMARY
Maximum and minimum principles for certain plate bending problems are derived in a unified manner from the canonical theory of complementary variational principles for multiple operator equations. The minimum principle is known in the literature, but the maximum principle appears to be new. A new error bound for approximate variational solutions is also presented.

1. Introduction

The boundary value problem described by the equations

$$L\phi \equiv \nabla^4\phi - \operatorname{div}(M \text{ grad } \phi) = f(x, y) \text{ in } R\,, \tag{1}$$

$$\phi = 0 \text{ on } \partial R\,, \tag{2}$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \partial R_1\,, \tag{3}$$

$$= b\nabla^2\phi \text{ on } \partial R_2 = \partial R - \partial R_1\,, \tag{4}$$

arises in the theory of thin elastic plates (cf. [1]). Here R is a region in the xy-plane which has a piecewise smooth boundary ∂R; ϕ is the deflection of the plate normal to the surface, $f(x, y)$ is a measure of normal loading, and M is a positive symmetric stress matrix. The boundary conditions (3) and (4) correspond to a plate which is clamped on parts ∂R_1 of the boundary and simply supported on the remainder $\partial R_2 = \partial R - \partial R_1$, where n is the outward pointing normal to the boundary, and $b = \rho/(1-\nu)$, ρ being the radius of curvature of ∂R and ν being Poisson's ratio. This includes the separate cases where the whole of the boundary is clamped ($\partial R_2 \equiv 0$) and where the whole of the boundary is simply supported ($\partial R_1 \equiv 0$).

A minimum principle associated with this class of problems is known in the literature [1]. In this paper we present a new complementary maximum principle together with a new error bound on variational solutions. The error bound is important since it provides an estimate of the accuracy of approximate solutions for the deflection of the plate.

2. Canonical formalism

The basic equations in the theory of complementary variational principles are the generalized canonical equations (see [2], [3])

$$T\phi = \frac{\partial H}{\partial u}\,, \tag{5}$$

$$T^* u = \frac{\partial H}{\partial \phi}\,, \tag{6}$$

where T is some linear operator and T^* is its adjoint with respect to a suitable inner product. To cast equation (1) into this form we write the positive symmetric matrix M as

$$M = N^t N \,, \tag{7}$$

where t denotes transpose, and introduce the operators

$$T_1 = \nabla^2 \,, \tag{8}$$

and

$$T_2 = N \text{ grad} \,, \tag{9}$$

and their adjoints

$$T_1^* = \nabla^2 \,, \tag{10}$$

$$T_2^* = -\operatorname{div}(N^t) \,, \tag{11}$$

where

$$\int_R \psi T_1 \phi \, dx\, dy = \int_R (T_1^* \psi) \phi \, dx\, dy + \text{boundary terms} \,, \tag{12}$$

$$\int_R w^t T_2 \phi \, dx\, dy = \int_R (T_2^* w) \phi \, dx\, dy + \text{boundary terms} \,, \tag{13}$$

for all suitable ψ, ϕ and w. These are now used to define the operators T and its adjoint T^* which are such that

$$T\phi = \begin{pmatrix} T_1 \phi \\ T_2 \phi \end{pmatrix} \tag{14}$$

for all suitable scalar functions ϕ, and

$$T^* u = T_1^* u_1 + T_2^* u_2 \tag{15}$$

for all suitable vector functions u with components u_1 and u_2, where u_2 itself is a 2-vector. From these definitions it follows directly that the relation between T and T^* can be written

$$\int_R u^t T\phi \, dx\, dy = \int_R (T^* u) \phi \, dx\, dy + B(u, \phi)_{\partial R} \,, \tag{16}$$

where

$$B(u, \phi)_{\partial R} = \int_{\partial R} \left(u_1 \frac{\partial \phi}{\partial n} - \frac{\partial u_1}{\partial n} \phi \right) ds + \int_{\partial R} \phi u_2^t N \begin{pmatrix} dy \\ -dx \end{pmatrix} . \tag{17}$$

From (1) and (7) to (11) we see that

$$L = T_1^* T_1 + T_2^* T_2 \tag{18}$$

$$= T^* T \tag{19}$$

by (14) and (15). Thus equation (1) takes the form

$$T^* T\phi = f(x, y) \text{ in } R \tag{20}$$

subject to the boundary conditions

$$\phi = 0 \text{ on } \partial R \tag{21}$$

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } \partial R_1 \tag{22}$$

$$= b T_1 \phi \text{ on } \partial R_2 \,. \tag{23}$$

We can write (20) in the canonical form of (5) and (6) by taking

$$T\phi = u = \frac{\partial H}{\partial u} \text{ in } R \tag{24}$$

$$T^* u = f(x, y) = \frac{\partial H}{\partial \phi} \text{ in } R \tag{25}$$

with boundary conditions

$$\phi = 0 \text{ on } \partial R \tag{26}$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \partial R_1 \tag{27}$$

$$= bu_1 \text{ on } \partial R_2 \tag{28}$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. A suitable Hamiltonian H in (24) and (25) is

$$H(u, \phi) = \tfrac{1}{2}u^t u + f\phi \,. \tag{29}$$

This completes the canonical formalism. A similar multiple operator formalism has recently been employed for a class of ordinary differential equations [4].

3. Complementary variational principles

To obtain variational principles associated with the boundary problem in (1) to (4) we introduce the functional (see [2], [3])

$$I(U, \Phi) = \int_R \{U^t T\Phi - H(U, \Phi)\}\, dx\, dy + [\text{boundary terms}]\,, \tag{30}$$

where the boundary terms are chosen to lead to the boundary conditions (2) to (4), or (26) to (28). In the present case this functional takes the form

$$\begin{aligned} I(U, \Phi) = &\int_R \{U^t T\Phi - H(U, \Phi)\}\, dx\, dy - \int_{\partial R} \left\{U_1 \frac{\partial\Phi}{\partial n} - \frac{\partial U_1}{\partial n}\Phi\right\} ds \\ &+ \int_{\partial R_2} \tfrac{1}{2}bU_1^2 ds - \int_{\partial R} \Phi U_2^t N \begin{pmatrix} dy \\ -dx \end{pmatrix}, \end{aligned} \tag{31}$$

$$= \int_R \{(T^* U)\Phi - H(U, \Phi)\}\, dx\, dy + \int_{\partial R_2} \tfrac{1}{2}bU_1^2 ds\,, \tag{32}$$

where we have used (16) and (17).

Let u and ϕ denote the exact solution pair of (24) to (28). Then the following results are readily verified.

3a. *First variational principle.* For arbitrary independent functions U, Φ the functional $I(U, \Phi)$ is stationary at (u, ϕ), the solution pair of the boundary value problem (24) to (28).
3b. *Second variational principle.* Let Φ be an admissible function which satisfies the boundary conditions

$$\Phi = 0 \text{ on } \partial R\,, \quad \frac{\partial\Phi}{\partial n} = 0 \text{ on } \partial R_1\,. \tag{33}$$

Then using (31) we define a functional $J(\Phi)$ by

$$J(\Phi) = I(U(\Phi), \Phi)\,, \quad U(\Phi) = T\Phi \text{ in } R\,, \quad U_1(\Phi) = \frac{1}{b}\frac{\partial\Phi}{\partial n} \text{ on } \partial R_2\,. \tag{34}$$

This gives

$$J(\Phi) = \int_R \{\tfrac{1}{2}(\nabla^2\Phi)^2 + \tfrac{1}{2}(\text{grad }\Phi)^t M(\text{grad }\Phi) - f\Phi\}\, dx\, dy - \int_{\partial R_2} \frac{1}{2b}\left(\frac{\partial\Phi}{\partial n}\right)^2 ds\,. \tag{35}$$

If we expand about ϕ we find that

$$J(\Phi) = I(u, \phi) + \delta^2 J\,, \tag{36}$$

where

$$\delta^2 J = \frac{1}{2}\int_R \{(\nabla^2\xi)^2 + (\text{grad }\xi)^t M(\text{grad }\xi)\}\, dx\, dy - \frac{1}{2}\int_{\partial R_2} \frac{1}{b}\left(\frac{\partial\xi}{\partial n}\right)^2 ds \tag{37}$$

is the second variation, with $\xi = \Phi - \phi$. From (36) we see that $J(\Phi)$ is stationary at ϕ.

3c. *Third variational principle.* Let U be an admissible function which satisfies the condition

$$T^*U = f \text{ in } R. \tag{38}$$

Using (32) we define a functional $G(U)$ by

$$G(U) = I(U, \Phi), \text{ [} U \text{ subject to (38)]}. \tag{39}$$

This gives

$$G(U) = -\tfrac{1}{2}\int_R U^t U \, dx\, dy + \tfrac{1}{2}\int_{\partial R_2} bU_1^2 \, ds. \tag{40}$$

If we expand about u it follows that

$$G(U) = I(u, \phi) + \delta^2 G, \tag{41}$$

where

$$\delta^2 G = -\tfrac{1}{2}\int_R (U-u)^t(U-u)\, dx\, dy + \tfrac{1}{2}\int_{\partial R_2} b(U_1 - u_1)^2 \, ds \tag{42}$$

is the second variation. From (41) we see that $G(U)$ is stationary at u.

Since the exact solution u is related to ϕ by $u = T\phi$ in R and $u_1 = b^{-1}(\partial\phi/\partial n)$ on ∂R_2, it is desirable to choose the function U to have the form

$$U = T\Psi \text{ in } R, \quad U_1 = \frac{1}{b}\frac{\partial \Psi}{\partial n} \text{ on } \partial R_2, \tag{43}$$

where Ψ is intended to be an approximation to ϕ. Then from (40)

$$G(U(\Psi)) = -\tfrac{1}{2}\int_R \{(\nabla^2\Psi)^2 + (\text{grad } \Psi)^t M (\text{grad } \Psi)\, dx\, dy + \tfrac{1}{2}\int_{\partial R_2} \frac{1}{b}\left(\frac{\partial \Psi}{\partial n}\right)^2 ds, \tag{44}$$

and here, by (38), the function Ψ satisfies the constraint

$$L\Psi = f \text{ in } R. \tag{45}$$

3d. *Minimum principle.* If $b < 0$ we see from (37) that $\delta^2 J$ is nonnegative. Hence, by (36), we obtain the minimum principle

$$I(u, \phi) \leqq J(\Phi), \tag{46}$$

where $J(\Phi)$ is given by (35) with Φ subject to (33), equality holding when Φ is equal to ϕ. This is the principle of minimum energy [1].

3e. *Maximum principle.* If $b < 0$ we see from (42) that $\delta^2 G$ is nonpositive. Hence, by (41), we obtain the maximum principle

$$G(U(\Psi)) \leqq I(u, \phi), \tag{47}$$

where $G(U(\Psi))$ is given by (44) with Ψ subject to (45), equality holding when Ψ is equal to ϕ. This maximum principle appears to be new.

3f. *Complementary variational principles.* Combining the results of 3d and 3e, we see that if $b < 0$ we have the complementary principles

$$G(U(\Psi)) \leqq I(u, \phi) \leqq J(\Phi), \tag{48}$$

where $J(\Phi)$ is given by (35) with Φ subject to (33), and $G(U(\Psi))$ is given by (44) with Ψ subject to (45).

Thus we have obtained upper and lower bounds for the quantity $I(u, \phi)$ which from (31) and (32) is

$$I(u, \phi) = \int_R \{\tfrac{1}{2}(\nabla^2\phi)^2 + \tfrac{1}{2}(\text{grad }\phi)^t M(\text{grad }\phi) - f\phi\}\,dx\,dy - \tfrac{1}{2}\int_{\partial R_2} \frac{1}{b}\left(\frac{\partial\phi}{\partial n}\right)^2 ds$$

$$= -\tfrac{1}{2}\int_R u^t u\,dx\,dy + \tfrac{1}{2}\int_{\partial R_2} b u_1^2\,ds\,.$$

These extremum principles can of course be used to generate approximate solutions of the boundary value problem by the Rayleigh–Ritz and other methods. Now if attention is centred on the variational solution Φ of the boundary value problem, it is desirable to have an estimate of the error in this approximate solution. When the complementary principles (48) hold, such an estimate, or error bound, can be obtained.

4. Error bound

We shall suppose that $b<0$ so that the complementary variational principles (48) hold. Then we can say that for any admissible functions Φ and Ψ

$$J(\Phi) - G(U(\Psi)) \geqq J(\Phi) - I(u, \phi)$$
$$= \delta^2 J\,, \tag{49}$$

where, by (37),

$$\delta^2 J = \tfrac{1}{2}\int_R (T\xi)^t(T\xi)\,dx\,dy - \tfrac{1}{2}\int_{\partial R_2} \frac{1}{b}\left(\frac{\partial\xi}{\partial n}\right)^2 ds\,, \tag{50}$$

with $\xi = \Phi - \phi$. Since Φ and ϕ satisfy conditions (33), we have

$$\xi = 0 \text{ on } \partial R\,, \quad \frac{\partial\xi}{\partial n} = 0 \text{ on } \partial R_1\,. \tag{51}$$

Integrating the first term on the right of (50) by parts, as in (16) and (17), and using (51) we obtain

$$\delta^2 J = \tfrac{1}{2}\int_R \xi L\xi\,dx\,dy + \tfrac{1}{2}\int_{\partial R_2}\left(\nabla^2\xi - \frac{1}{b}\frac{\partial\xi}{\partial n}\right)\frac{\partial\xi}{\partial n}\,ds\,. \tag{52}$$

To get a useful result from this we make the integral over ∂R_2 in (52) vanish. Thus we shall require

$$\frac{\partial\xi}{\partial n} = b\nabla^2\xi \text{ on } \partial R_2\,, \tag{53}$$

and since $\xi = \Phi - \phi$ and ϕ satisfies (4), equation (53) implies that

$$\frac{\partial\Phi}{\partial n} = b\nabla^2\Phi \text{ on } \partial R_2\,. \tag{54}$$

This means that the function Φ satisfies all of the boundary conditions (2) to (4). Then (52) reduces to

$$\delta^2 J = \tfrac{1}{2}\int_R \xi L\xi\,dx\,dy\,. \tag{55}$$

If Λ is a lower bound to the (positive) lowest eigenvalue of

$$L\theta = \lambda\theta \text{ in } R\,, \tag{56}$$

subject to

$$\theta = 0 \text{ on } \partial R\,, \tag{57}$$

$$\frac{\partial\theta}{\partial n} = 0 \text{ on } \partial R_1\,, \tag{58}$$

$$= b\nabla^2\theta \text{ on } \partial R_2\,, \tag{59}$$

it follows that

$$\delta^2 J \geqq \tfrac{1}{2}\Lambda \|\Phi-\phi\|_{L^2}^2 , \tag{60}$$

where $\|\psi\|_{L^2}$ denotes the L^2 norm $\{\int_R \psi^2 dx\,dy\}^{\frac{1}{2}}$. From (60) and (49) we therefore have the error bound

$$\|\Phi-\phi\|_{L^2} \leqq [\{2J(\Phi)-2G(U(\Psi))\}\Lambda^{-1}]^{\frac{1}{2}} = E(\Phi) \text{ say} . \tag{61}$$

In expression (61), $J(\Phi)$ is given by (35) and the variational function Φ must satisfy all of the given boundary conditions (2) to (4), $G(U(\Psi))$ is given by (44) and Ψ must satisfy condition (45), and Λ is obtained from the eigenproblem in (56) to (59).

5. An example

To illustrate these results we consider the problem

$$L\phi \equiv \nabla^4\phi - \nabla^2\phi = 1 \text{ in } R , \tag{62}$$

$$\phi = 0 \text{ on } \partial R , \tag{63}$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \partial R , \tag{64}$$

for a clamped plate. This corresponds to

$$f(x, y) = 1 , \quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{65}$$

$$\partial R_1 = \partial R , \quad \partial R_2 = 0 , \tag{66}$$

and we take R to be the square

$$R = \{-1 \leqq x, y \leqq 1\} . \tag{67}$$

We have performed calculations with trial functions of the form

$$\Phi = (1-x^2)^2(1-y^2)^2\{a_1 + a_2(x^2+y^2) + a_3x^2y^2 + a_4(x^4+y^4)\} , \tag{68}$$

and

$$\Psi = -\tfrac{1}{4}(x^2+y^2) + b_1 \cosh\frac{x}{\sqrt{2}}\cosh\frac{y}{\sqrt{2}} + b_2(\cosh x + \cosh y) + \\ + \sum_{r=3}^{5} b_r(\cosh px \cos py + \cos px \cosh py) , \tag{69}$$

with $p=(r-\frac{5}{2})\pi$. The function Φ in (68) satisfies the exact boundary conditions (63) and (64), and the function Ψ in (69) satisfies the constraint (45), that is $L\Psi = 1$ in R. The parameters a_r and b_r were determined by optimizing the functionals J and G, and the results including the error bound are given in Table 1. In this example, the lowest eigenvalue of the problem in (56)

TABLE 1

Variational parameters and error bound

a_1	a_2	a_3	a_4	J	E
1.8789 (−2)	5.3218 (−3)	6.7401 (−3)	3.9789 (−4)	−1.165338 (−2)	2.3 (−4)

b_1	b_2	b_3	b_4	b_5	G
−8.0044	4.4378	−1.8817 (−1)	5.6447 (−5)	−2.0251 (−7)	−1.165557 (−2)

Here $m(-n)$ means $m \times 10^{-n}$.

to (59) has a lower bound given by $\Lambda = 80.86$ ([5]). It can be seen from Table 1 that the variational solution (68) is quite accurate, its maximum error being 2.3×10^{-4}.

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